Intermittency and exponent field dynamics in developed turbulence

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Spatiotemporal dynamics of intermittency in association with coarse-grained energy-dissipation rate fluctuations is discussed. This is done first by phenomenologically constructing the probability density for exponent field fluctuations that is introduced to characterize the energy-dissipation rate field, and then by proposing the Langevin dynamics derived with the projection-operator method on the basis of the Navier-Stokes equation. With a Gaussian approximation for exponent fluctuations, spatiotemporal correlation functions for coarse-grained energy-dissipation rate fluctuations are explicitly obtained.

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I. INTRODUCTION

The intermittency effect on the energy spectrum [1] and the velocity structure functions as well as the energy-dissipation structure functions due to the coexistence of turbulent and less turbulent regions in fluid is one of the main subjects of studies of isotropic homogeneous turbulence not only in developed turbulence [2,3] but also in the intermediate Reynolds number turbulence [4–7]. As formulated first by Kolmogorov [8] and Obukhov [9], it has been believed that the intermittency comes in due to small scale fluctuations of the coarse-grained energy-dissipation rate

$$\epsilon_r(\boldsymbol{x})=\frac{1}{4\pi r^3/3}\int_{|\boldsymbol{r}|<r}\epsilon_{\rm local}(\boldsymbol{x}+\boldsymbol{r})d\boldsymbol{r}, \tag{1}$$

where

$$\epsilon_{\rm local}(\boldsymbol{x})=\frac{\nu}{2}\sum_{i,j}\left(\frac{\partial v_i(\boldsymbol{x})}{\partial x_j}+\frac{\partial v_j(\boldsymbol{x})}{\partial x_i}\right)^2 \tag{2}$$

is the local energy-dissipation rate per mass. The $v_i(\boldsymbol{x})$ is the ith component of the velocity field at a given time, and ν is the kinematic viscosity.

In the inertial subrange in between the Kolmogorov microscale η and the energy injection scale L, the velocity structure function obeys the power law $\psi_q^u(r)\equiv\langle u_r^q\rangle\sim r^{\zeta(q)}$ in a high Reynolds number (developed) turbulence $L/\eta\to\infty$, where $u_r=|[\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r})-\boldsymbol{v}(\boldsymbol{x})]\cdot\boldsymbol{r}/r|$ is the longitudinal component of the velocity difference at locations separated by the distance r chosen as $\eta<r<L$ [2,3]. The brackets $\langle\cdots\rangle$ stand for the average over an ensemble describing the steady turbulent state. Similarly, the coarse-grained energy-dissipation rate structure function satisfies the power law $\psi_q^\epsilon(r)\equiv\langle\epsilon_r^q\rangle\sim r^{\tau(q)}$ in the inertial subrange [2,3]. The refined self-similarity hypothesis [8,9] and the multifractal interpretation [10] leads to the relation $\zeta(q)=q/3+\tau(q/3)$, although it is not confirmed yet. If there exists no relevant fluctuation in ϵ_r, no intermittency is present. So it is quite natural to use coarse-grained energy-dissipation rate fluctuations themselves to study intermittency in turbulence.

The intermittency conventionally discussed is thus associated with spatial fluctuations of the coarse-grained energy-dissipation rate. More precisely speaking, with the homogeneity assumption of turbulence, its statistics might be discussed for fluctuation at one position with the help of the time average of the fluctuation. However, in addition to that, the coarse-grained energy-dissipation rate is the field variable defined for position $\boldsymbol{x}$, around which the averaging is carried out, the velocity field temporally changes according to the Navier-Stokes equation. This fact implies that the coarse-grained energy-dissipation rate $\epsilon_r(\boldsymbol{x},t)$ is a random variable depending on both position and time. Therefore, in order to study the global statistics of intermittency, spatiotemporal statistics of the energy-dissipation rate fluctuations have to be studied. This is the fundamental motivation of the present study. For this aim, in the present paper, we will phenomenologically construct the probability density for energy-dissipation rate field and develop an approach to characterize the field dynamics of intermittency from a nonequilibrium statistical-physical viewpoint.

The paper is organized as follows. In Sec. II, we briefly discuss the intermittency on the energy-dissipation rate fluctuation from the self-similarity viewpoint, and find that its asymptotic probability density is characterized by the fluctuation spectrum $S(z)$. Then we will phenomenologically introduce the probability density for fluctuations of energy-dissipation rate field, introducing the exponent field that explicitly describes the self-similarity of the energy-dissipation rate field. In Sec. III, we will derive the Langevin equations and the corresponding Fokker-Planck equations for exponent fluctuations and energy-dissipation rate fluctuations, starting with the Navier-Stokes equation with the aid of the projection-operator method developed in the nonequilibrium statistical mechanics [11]. In Sec. IV, the spatiotemporal correlation functions for exponent fluctuations and energy-dissipation rate fluctuations are explicitly obtained with the parabolic approximation for $S(z)$. Concluding remarks are given in Sec. V. Appendix A is devoted to the details of the derivation of the Langevin equation for exponent field. The eigenvalue problem to derive temporal corre-

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lations of exponent fluctuations and energy-dissipation rate fluctuations for spatially uniform fluctuations is given in Appendix B.

II. SELF-SIMILARITY STATISTICS OF ENERGY-DISSIPATION RATE FLUCTUATIONS

A. Probability density for the energy-dissipation rate

In this section we will first briefly review how to discuss the intermittency in the statistics of the coarse-grained energy-dissipation rate fluctuation $\epsilon_r(\boldsymbol{x},t)$ [6–9]. Let us define discrete scales by

$$r_n = L b^{-n} \quad (n=0,1,2,\ldots,N), \tag{3}$$

where $b(>1)$ is an arbitrary positive constant, and the maximum step $N=\log_b(L/\eta)$ is associated with the Kolmogorov scale η, the small cutoff of the inertial subrange, and is sufficiently large due to the high Reynolds number turbulence.

We introduce exponents $\{z_n\}$ by

$$\frac{\epsilon_{r_{n+1}}}{\epsilon_{r_n}} = b^{z_n}. \tag{4}$$

If z_n vanishes for any n, then the coarse-grained energy-dissipation rate shows no fluctuation, which corresponds to the Kolmogorov theory in 1941 [1]. However, exponents $\{z_n\}$ are in general random variables. This fact is believed to be the origin of intermittency. Explicit statistical characteristics of $\{z_n\}$ are defined as follows. We assume that (i) the z_n's obey the same statistics for all n, (ii) do not have any statistical anomaly such as divergent variance, and that (iii) the correlation function, $\langle \delta z_{i+j} \delta z_j \rangle, (\delta z_j = z_j - \langle z \rangle)$, decays sufficiently rapidly in comparison with N. These assumptions are based on the self-similarity of the energy cascade process [8] since the assumed statistical characteristics of the ratio indices z_n's are free from n indicating the scale and, therefore, are free from the scale r_n itself. Equation (4) is integrated to yield

$$\epsilon_{r_n} = \epsilon_L \left(\frac{L}{r_n} \right)^{\overline{z_n}}, \tag{5}$$

where

$$\overline{z_n} = \frac{1}{n} \sum_{j=0}^{n-1} z_j . \tag{6}$$

Here, ϵ_L is the coarse-grained energy-dissipation rate averaged over the energy injection scale L. We, hereafter, assume that fluctuation of ϵ_L is small, and that ϵ_L is a constant.

Let us introduce the probability density $Q_{r_n}(z)$ for $\overline{z_n}$. Since $\overline{z_n}$ is the sum of random variables whose correlation step is sufficiently short, it is expected that $Q_{r_n}(z)$ satisfies the asymptotic form

$$Q_{r_n}(z) \sim b^{-S(z)n} = \left(\frac{L}{r_n} \right)^{-S(z)} \tag{7}$$

for large n, where the function $S(z)$ is independent of n and b, and fulfills the conditions

$$S(z) \geqslant 0, \quad S''(z) > 0. \tag{8}$$

It is expected that $S(z)$ is a universal function that characterizes the self-similarity of energy cascade process in developed turbulence. The asymptotic form (7) with Eq. (8) is the consequence of the large deviation theory [12–14] in the probability theory. The concavity condition of $S(z)$ implies the existence of only one minimum at $z=z_0$, where $S(z_0)$ vanishes. The log normal theory of intermittency [8,9] is based on the central limit theorem and assumes the parabolic form for $S(z)$.

The above asymptotic probability density thus leads to the probability density $P_r(\epsilon)$ for ϵ_r takes the asymptotic form [6,7,15]

$$P_r(\epsilon) \sim \epsilon^{-1} \left(\frac{L}{r} \right)^{-S(z_r(\epsilon))} \sim \left(\frac{L}{r} \right)^{-S(z_r(\epsilon)) - z_r(\epsilon)}, \tag{9}$$

$$z_r(\epsilon) \equiv \frac{\ln \dfrac{\epsilon}{\epsilon_L}}{\ln \dfrac{L}{r}}. \tag{10}$$

Alternatively, the structure function $\psi_q^{\epsilon}(r) = \langle \epsilon_r^q \rangle$ obeys the asymptotic law

$$\psi_q^{\epsilon}(r) \sim r^{\tau(q)} \tag{11}$$

for $\eta \ll r \ll L$, where the characteristic functions $\tau(q)$ and $S(z)$ are related to each other via the Legendre transform,

$$\tau(q) = \min_z [S(z) - qz]. \tag{12}$$

It should be noted that the asymptotic form (9) can be alternatively derived as follows. Let $P(\epsilon, r | \epsilon', r')$ $(r < r')$ be the conditional probability density, in which ϵ_r takes the value ϵ in the region where $\epsilon_{r'}$ takes the value ϵ', and r and r' are arbitrarily chosen in the inertial subrange. The self-similarity in the inertial subrange is formulated to satisfy the chain relation [7]

$$P(\epsilon_3, r_3 | \epsilon_1, r_1) = \int P(\epsilon_3, r_3 | \epsilon_2, r_2) P(\epsilon_2, r_2 | \epsilon_1, r_1) d\epsilon_2, \tag{13}$$

for $\eta \ll r_3 \ll r_2 \ll r_1 \ll L$. This equation has a structure similar to that determining the steady state solution of the Chapman-Kolmogorov equation in the Markov process. The above three probability densities should take same asymptotic forms for $\eta \ll r_i \ll L$, $(i=1,2,3)$. This is the explicit mathematical expression of self-similarity of the energy-dissipation rate fluctuations. As shown in Ref. [7], this equation yields the solution

$$P(\epsilon,r|\epsilon',r')\sim\epsilon^{-1}\left(\frac{r'}{r}\right)^{-S(z(\epsilon,r|\epsilon',r'))}, \tag{14}$$

$$z(\epsilon,r|\epsilon',r')=\frac{\ln\frac{\epsilon}{\epsilon'}}{\ln\frac{r'}{r}}, \tag{15}$$

where $S(z)$ is a concave function of z. Particularly, assuming that no fluctuation is present in ϵ_L and putting $P_r(\epsilon)=P(\epsilon,r|\epsilon_L,L)$, we find that the formula (14) is reduced to Eq. (9).

The above consideration in deriving the probability density $P_r(\epsilon)$ raises two possibilities of the extension of the above formulation to study the energy-dissipation rate fluctuations. This is related with the problems on (i) how to discuss the spatial fluctuations of $\epsilon_r(\boldsymbol{x})$ and (ii) how to derive the temporal evolution of coarse-grained energy-dissipation rate fluctuations. The second problem will be addressed in Sec. III. In the remaining part of this section, we will phenomenologically discuss a possible way to take into account the spatial fluctuations of $\epsilon_r(\boldsymbol{x},t)$ for a given time, where the time evolution of $\epsilon(\boldsymbol{x},t)$ is generated by that of the velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ in Eq. (2).

B. Generalization to fluctuations of the energy-dissipation rate field

As one observes that the coarse-grained energy-dissipation rate $\epsilon_r(\boldsymbol{x})$ depends on the position $\boldsymbol{x}$, which is the center of the coarse-graining procedure. Let us discuss the probability density of the energy-dissipation rate fluctuations for the whole system. In order to do that, we first define the probability density

$$P_r\{\epsilon\}\equiv\left\langle\prod_{\boldsymbol{x}}\delta(\epsilon_r(\boldsymbol{x})-\epsilon(\boldsymbol{x}))\right\rangle \tag{16}$$

for the fluctuation field $\{\epsilon_r(\boldsymbol{x})\}$, where $\{\epsilon(\boldsymbol{x})\}$ is the value of $\epsilon_r(\boldsymbol{x})$ and $\{\epsilon\}$ stands for the set of $\epsilon(\boldsymbol{x})$ for the whole space. Here, we defined the product $\Pi_{\boldsymbol{x}}$ by

$$\prod_{\boldsymbol{x}}A(\boldsymbol{x})\equiv\exp\left[a^{-3}\int\ln A(\boldsymbol{x})d\boldsymbol{x}\right], \tag{17}$$

with $a(\ll\eta)$ being a quantity which has the dimension of length. Without loss of generality, hereafter we put $a=1$. The brackets $\langle\cdots\rangle$ imply the average over a suitable steady-state turbulence ensemble.

The asymptotic form of $P_r\{\epsilon\}$ in terms of the single point probability density $P_r(\epsilon)$ may be constructed as follows. First, let us introduce the exponent field $\overline{z_r}(\boldsymbol{x})$ by

$$\epsilon_r(\boldsymbol{x})=\epsilon_L\left(\frac{L}{r}\right)^{\overline{z_r}(\boldsymbol{x})},\quad \overline{z_r}(\boldsymbol{x})=\frac{\ln\frac{\epsilon_r(\boldsymbol{x})}{\epsilon_L}}{\ln\frac{L}{r}}. \tag{18}$$

Although the energy-dissipation rate field $\epsilon_r(\boldsymbol{x})$ is anomalous in the sense that its statistics shows a power law behavior in r, the exponent field $\overline{z_r}(\boldsymbol{x})$ is expected to have no anomalous statistics. This is the fundamental hypothesis being compatible with that made in the preceding section to derive the asymptotic form (9). The probability density for the exponent field defined as

$$Q_r\{z\}\equiv\left\langle\prod_{\boldsymbol{x}}\delta(\overline{z_r}(\boldsymbol{x})-z(\boldsymbol{x}))\right\rangle \tag{19}$$

is related to the probability density $P_r\{\epsilon\}$ via

$$P_r\{\epsilon\}=Q_r\left\{\frac{\ln\frac{\epsilon}{\epsilon_L}}{\ln\frac{L}{r}}\right\}\prod_{\boldsymbol{x}}\left\{\epsilon(\boldsymbol{x})\ln\frac{L}{r}\right\}^{-1}. \tag{20}$$

It is expected that $\ln Q_r\{z\}$ is extensive with respect to space, i.e., $\ln Q_r\{z\}$ is the quantity of the order of the system volume that is chosen to be sufficiently large. This is the consequence of the assumption that the exponent field is expected to be nonanomalous in both space and time. Furthermore, if there exist inhomogeneous fluctuations in exponent field in homogeneous turbulence, they may reduce the probability density $\sim\Pi_{\boldsymbol{x}}(L/r)^{-S(z(\boldsymbol{x}))}=(L/r)^{-\int S(z(\boldsymbol{x}))d\boldsymbol{x}}$, which is the correct expression if fluctuations in each position were independent of each other. However, since the velocity field in a turbulent fluid has a spatial correlation at a given time, the energy-dissiparion rate field and, therefore, the exponent field has a spatial correlation. This fact implies that the probability density for the exponent field cannot be given by the simple product of probability densities at all positions. The correlation structure is uniquely determined by the Navier-Stokes equation. However, we do not know details of the correlation behavior. So, we try to phenomenologically construct the probability density for the exponent field.

First, remember that we are now considering a homogeneous, isotropic turbulence, where if a spatial variation of exponent field appears in a some region at a given time, then its spatial structure always tends to decay to a locally homogeneous exponent field. One should note that this argument does not mean that the exponent field ultimately tends to a homogeneous state. Instead, a strong chaotic nature of turbulence incessantly creates a local inhomogeneity of exponent field. This consideration may be expressed in such a way that the true probability density for exponent field is smaller than the simple product of probability densities for the whole space because of the presence of spatial inhomogeneity. The next problem is how to mathematically formulate the reduction of the probability density for exponent field in the presence of the inhomogeneity of exponent field. To this aim, here we borrow the idea of the Landau theory of thermodynamic critical phenomena, particularly for the ferromagnetic Ising spin system, where the reduction of the probability density for order parameter field in the presence of its spatial variation is expressed by adding a term given by the gradient of order parameter in the Landau Hamiltonian. In this way, to take into account the inhomogeneity of exponent field, as the simplified approximation, we propose the form

$$Q_r\{z\} \propto \left(\frac{L}{r}\right)^{-\mathcal{H}\{z\}}, \tag{21}$$

with

$$\mathcal{H}\{z\} = \int \left[S(z(\boldsymbol{x})) + \frac{c_r}{2}(\nabla_{\boldsymbol{x}} z(\boldsymbol{x}))^2 \right] d\boldsymbol{x}, \tag{22}$$

where c_r is a positive constant, which might depend on the coarse-graining scale r, and measures the stiffness of the homogeneity of exponent field in the turbulence under consideration. The presence of the $(\nabla_{\boldsymbol{x}} z)^2$ is crucial to the reduction of the probability in the presence of spatial inhomogeneity of exponent field. The above form is the simplest one to take into account the fluctuations of exponent field. With Eq. (21), the probability density (16) is written as

$$P_r\{\epsilon\} \sim \left(\frac{L}{r}\right)^{-\mathcal{F}\{\ln(\epsilon/\epsilon_L)/\ln(L/r)\}}, \tag{23}$$

$$\mathcal{F}\{z\} = \mathcal{H}\{z\} + \int z(\boldsymbol{x}) d\boldsymbol{x}. \tag{24}$$

This is one of the fundamental proposals of the present paper. The stochastic dynamics should be constructed so as to yield the steady probability densities (21) and (23).

III. DERIVATION OF THE LANGEVIN EQUATION FOR AN EXPONENT FIELD BASED ON THE HYDRODYNAMIC EQUATION

Let us consider the 3D Navier-Stokes equation for incompressible fluid with the external force,

$$\frac{\partial \boldsymbol{v}(\boldsymbol{x},t)}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}(\boldsymbol{x},t) = -\boldsymbol{\nabla} p(\boldsymbol{x},t) + \nu \nabla^2 \boldsymbol{v}(\boldsymbol{x},t) + \boldsymbol{f}_{\mathrm{ext}}(\boldsymbol{x}), \tag{25}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{x},t) = 0, \tag{26}$$

where $\boldsymbol{v}(\boldsymbol{x},t)$ is the velocity field at time t, and $p(\boldsymbol{x},t)$ is the pressure field. Hereafter, the external force $\boldsymbol{f}_{\mathrm{ext}}(\boldsymbol{x})$ is assumed to be time independent, and its characteristic length of spatial variation is L. Namely, the spatial power spectrum $I_{\boldsymbol{k}} = \frac{1}{3}\sum_{\alpha=x,y,z}\langle|\int f_{\mathrm{ext},\alpha}(\boldsymbol{x})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}d\boldsymbol{x}|^2\rangle$ of $\boldsymbol{f}_{\mathrm{ext}}(\boldsymbol{x})$ has significant magnitude at the wave number $\boldsymbol{k}_L$ with $|\boldsymbol{k}_L| = 2\pi/L$. The simplest form of the external field is $\boldsymbol{f}_{\mathrm{ext}}(\boldsymbol{x}) = \boldsymbol{A}\cos(\boldsymbol{k}_L\cdot\boldsymbol{x} + \theta_0)$ with a constant magnitude vector $\boldsymbol{A}$ and a constant phase θ_0. However, the universal nature of developed turbulence suggests that the statistics in the inertial subrange does not depend on details of the mechanism of the external forcing. Therefore, hereafter, we do not need to impose further detail on the external field. The state vector composed of the whole components of coordinates and positions, defined by

$$\boldsymbol{X}(t) = \{v_x(\boldsymbol{x},t), v_y(\boldsymbol{x},t) v_z(\boldsymbol{x},t)\} = (X_1(t), X_2(t), X_3(t), \ldots) \tag{27}$$

contains the full microscopic informations of the velocity field at time t. In terms of the state vector $\boldsymbol{X}(t)$, the Navier-Stokes equation (25) is formally rewritten as the set of autonomous equations of motion

$$\dot{\boldsymbol{X}}(t) = \boldsymbol{F}(\boldsymbol{X}(t)). \tag{28}$$

Time evolution of any spatially coarse-grained variable is determined by this set of equations of motion. For an arbitrary initial condition, the equation of motion (28) has a bound solution in the state space.

The distribution

$$\delta_{\boldsymbol{X}}(t) \equiv \prod_j \delta(X_j(t) - X_j) \tag{29}$$

obeys the equation of motion

$$\frac{\partial \delta_{\boldsymbol{X}}(t)}{\partial t} = -\sum_j \frac{\partial}{\partial X_j}[F_j(\boldsymbol{X})\delta_{\boldsymbol{X}}(t)] \equiv H\delta_{\boldsymbol{X}}(t). \tag{30}$$

Let $G(t)$ be an arbitrary function of $\boldsymbol{X}(t)$, i.e., $G(t) = G\{\boldsymbol{X}(t)\}$. By noting

$$G(t) = \int \delta_{\boldsymbol{X}}(t) G\{\boldsymbol{X}\} d\boldsymbol{X} = \int \delta_{\boldsymbol{X}}(0) e^{tL(\boldsymbol{X})} G\{\boldsymbol{X}\} d\boldsymbol{X}, \tag{31}$$

the time evolution of $G(t)$ is determined by

$$\dot{G}(t) = LG(t), \tag{32}$$

with the linear differential operator

$$L = \sum_j F_j(\boldsymbol{X}) \frac{\partial}{\partial X_j}, \tag{33}$$

where $\boldsymbol{X} = \boldsymbol{X}(0)$. The operator L is adjoint to H, and satisfies the relation

$$L(G_1 G_2) = (LG_1)G_2 + G_1 L G_2 \tag{34}$$

for arbitrary functions G_1 and G_2 of $\boldsymbol{X}$.

By choosing

$$G(t) = \prod_{\boldsymbol{x}} \delta(\overline{z_r}(\boldsymbol{x},t) - z(\boldsymbol{x})) \equiv g_z(t), \tag{35}$$

where $\overline{z_r}(\boldsymbol{x},t)$ is defined via $\overline{z_r}(\boldsymbol{x},t) = \ln(\epsilon_r(\boldsymbol{x},t)/\epsilon_L)/\ln(L/r)$ with the coarse-grained energy-dissipation rate $\epsilon_r(\boldsymbol{x},t)$ defined with the velocity field $\boldsymbol{v}(\boldsymbol{x},t)$ at time t. The subscript z stands for the set $\{z(\boldsymbol{x})\}$. The $g_z(t)$ is thus found to obey

$$\frac{\partial g_z(t)}{\partial t} = Lg_z(t). \tag{36}$$

This is our starting equation to derive the Langevin dynamics for $\overline{z_r}(\boldsymbol{x},t)$ and $\epsilon_r(\boldsymbol{x},t)$.

We assume that the steady turbulence described by Eqs. (25) and (26) with an additional boundary condition is a steady spatiotemporal chaos, and that for the steady turbulence under consideration, the ergodicity holds. Namely, the ensemble average suitably chosen is equal to the long-time average uniquely determined for almost all initial conditions. As known in the theory of dynamical systems, no invariant density generally exists. By noting this fact, the average procedure in the following discussion will be done by using the time average instead of the ensemble average. Let us introduce the invariant measure $\mu(\boldsymbol{X})$ by

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t G\{\boldsymbol{X}(s)\}ds=\int G\{\boldsymbol{X}\}d\mu(\boldsymbol{X})\equiv\langle G\rangle, \tag{37}$$

which holds for almost all initial conditions $\boldsymbol{X}(0)$ and an arbitrary function $G\{\boldsymbol{X}(0)\}$, where $G\{\boldsymbol{X}(s)\}$ is finite for any time. Taking the long-time average of $dG\{\boldsymbol{X}(t)\}/dt = LG\{\boldsymbol{X}(t)\}$, one obtains

$$\int LG\{\boldsymbol{X}\}d\mu(\boldsymbol{X})=\langle LG\rangle=0. \tag{38}$$

The combination of Eqs. (38) and (34) yields

$$\langle[LG_1]G_2\rangle=-\langle G_1[LG_2]\rangle \tag{39}$$

for arbitrary finite functions G_1 and G_2 of $\boldsymbol{X}$.

In terms of the above definition of average, we define the projection operator $\mathcal{P}$ by

$$\mathcal{P}G\{\boldsymbol{X}\}=\int\langle G\{\boldsymbol{X}\};z\rangle g_z(0)dz=\langle G\{\boldsymbol{X}\};\overline{z(0)}\rangle, \tag{40}$$

where $\overline{z(t)}$ stands for the set $\{\overline{z(\boldsymbol{x},t)}\}$, and

$$\langle G\{\boldsymbol{X}\};z\rangle\equiv\int G\{\boldsymbol{X}\}\frac{g_z(0)}{Q_r^0\{z\}}d\mu(\boldsymbol{X})=\langle G\{\boldsymbol{X}\}g_z(0)\rangle/Q_r^0\{z\}, \tag{41}$$

with $Q_r^0\{z\}$ being the steady probability density (21) for exponent field. The projection operator $\mathcal{P}$ eliminates degrees of freedom except the exponent field. As shown in Appendix A, by making use of the above projection operator, the equation of motion (36) can be transformed into the following Langevin equation [11]:

$$\frac{\partial g_z(t)}{\partial t}=\mathcal{L}g_z(t)+F_z(t). \tag{42}$$

In deriving Eq. (42), we used three major approximations. For details, see below. Here, $\mathcal{L}$ is the linear operator defined by

$$\mathcal{L}G(z)=\int\frac{\delta}{\delta z(\boldsymbol{x})}\left[\Gamma_r(z(\boldsymbol{x}))Q_r^0\{z\}\frac{\delta}{\delta z(\boldsymbol{x})}\left\{\frac{G(z)}{Q_r^0\{z\}}\right\}\right]d\boldsymbol{x}, \tag{43}$$

and $F_z(t)$ is the Langevin random force defined by

$$F_z(t)=-\int\frac{\delta}{\delta z(\boldsymbol{x})}[R_r(\boldsymbol{x},t)g_z(0)]d\boldsymbol{x}, \tag{44}$$

where $\delta/\delta z(\boldsymbol{x})$ is the functional derivative. $R_r(\boldsymbol{x},t)$ is the random force appearing in the Langevin equation of $\overline{z(\boldsymbol{x},t)}$, [see Eq. (52)], and is assumed to be Gaussian white, satisfying

$$\langle R_r(\boldsymbol{x},t)\rangle=0, \tag{45}$$

$$\langle R_r(\boldsymbol{x},t)R_r(\boldsymbol{x}',0);z\rangle=2\Gamma_r(z)\delta(\boldsymbol{x}-\boldsymbol{x}')\delta(t), \tag{46}$$

i.e., the noise strength $\Gamma_r(z)$ is given by

$$\Gamma_r(z)=\int_0^\infty\int\langle R_r(\boldsymbol{x},t)R_r(\boldsymbol{x}',0);z\rangle d(\boldsymbol{x}-\boldsymbol{x}')dt. \tag{47}$$

The microscopic expression of $R_r(\boldsymbol{x},t)$ is given in Appendix A. In deriving Eq. (42), we have used three main approximations [16]. The first is that we assumed the absence of any collective motion of the exponent field. The second is that the linear operator $\mathcal{L}$ is terminated at the second order with respect to $\delta/\delta z$, which makes the master equation reduce to the Fokker-Planck equation. This is related to the Gaussian approximation for $R_r(\boldsymbol{x},t)$. Third, we assumed that the Langevin equation for the exponent field $\{\overline{z_r}(\boldsymbol{x},t)\}$ derived by eliminating other dynamical variables is Markoffian, i.e., it does not depend on exponent field in the past before t, $\{\overline{z_r}(\boldsymbol{x},s)\}(s<t)$. The Markovian approximation can be justified if the time scales of the exponent field and the Langevin random force $R_r(\boldsymbol{x})$ are sufficiently separated.

For the probability densities

$$Q_r\{z,t\}\equiv\langle g_z(t)\rangle, \tag{48}$$

$$P_r\{\epsilon,t\}\equiv\left\langle\prod_{\boldsymbol{x}}\delta(\epsilon_r(\boldsymbol{x},t)-\epsilon(\boldsymbol{x}))\right\rangle = Q_r\left\{\frac{\ln\frac{\epsilon}{\epsilon_L}}{\ln\frac{L}{r}},t\right\}\prod_{\boldsymbol{x}}\left\{\epsilon(\boldsymbol{x})\ln\frac{L}{r}\right\}^{-1}, \tag{49}$$

taking the ensemble average of Eq. (42), where the ensemble is same as the true turbulence ensemble, we get the Fokker-Planck equations

$$\frac{\partial Q_r\{z,t\}}{\partial t}=\int\frac{\delta}{\delta z(\boldsymbol{x})}\left[\Gamma_r(z(\boldsymbol{x}))Q_r^0\{z\}\frac{\delta}{\delta z(\boldsymbol{x})}\left\{\frac{Q_r\{z,t\}}{Q_r^0\{z\}}\right\}\right]d\boldsymbol{x}, \tag{50}$$

$$\frac{\partial P_r\{\epsilon,t\}}{\partial t}=\left(\ln\frac{L}{r}\right)^2\int\frac{\delta}{\delta\epsilon(\boldsymbol{x})}\left[\Gamma_r\left(\frac{\ln\frac{\epsilon(\boldsymbol{x})}{\epsilon_L}}{\ln\frac{L}{r}}\right)\times(\epsilon(\boldsymbol{x}))^2P_r^0\{\epsilon\}\frac{\delta}{\delta\epsilon(\boldsymbol{x})}\left\{\frac{P_r\{\epsilon,t\}}{P_r^0\{\epsilon\}}\right\}\right]d\boldsymbol{x}, \tag{51}$$

where $P_r^0\{\epsilon\}$ is the steady probability density (23) for energy-dissipation rate field. In deriving Eq. (51), we used the relation

$$\frac{\delta}{\delta\epsilon(\boldsymbol{x})}\left[\epsilon(\boldsymbol{x})\prod_{\boldsymbol{y}}\{\epsilon(\boldsymbol{y})\}^{-1}\right]=\frac{\delta}{\delta\epsilon(\boldsymbol{x})}[\epsilon(\boldsymbol{x})e^{-\int\ln\epsilon(\boldsymbol{y})d\boldsymbol{y}}]=0.$$

We have no solid criterion to determine the r dependence of the noise intensity Γ_r, although it is, in principle, determined by the Navier-Stokes dynamics. However, since we assume that the intermittency characteristic in the inertial subrange is completely determined by the statistics of $\bar{z}_r$, $R_r(\boldsymbol{x},t)$ belongs to the "microscopic" degrees of freedom in the sense of the projected-out part in the language of the projection-operator approach explained in Appendix A besides those of the inertial subrange. So it is natural to use the approximation that the damping constant $\Gamma_r(z)$ is independent of the "macroscopic" variable z. This consideration might conclude that $\Gamma_r(z)$ has no explicit dependence on both z and r. Hereafter, for simplicity, we assume that $\Gamma_r(z)$ is free from z.

From Eq. (42), we obtain the following Langevin equations:

$$\dot{\bar{z}}(\boldsymbol{x},t)=-\left(\Gamma_r\ln\frac{L}{r}\right)\frac{\delta\mathcal{H}\{\bar{z}_r\}}{\delta\bar{z}_r}+R_r(\boldsymbol{x},t), \tag{52}$$

$$\dot{\epsilon}_r(\boldsymbol{x},t)=\left(\ln\frac{L}{r}\right)\epsilon_r\left[-\left(\Gamma_r\ln\frac{L}{r}\right)\frac{\delta\mathcal{H}\{\bar{z}_r\}}{\delta\bar{z}_r}\bigg|_{\bar{z}_r=\ln(\epsilon_r/\epsilon_L)/\ln(L/r)}+R_r(\boldsymbol{x},t)\right]. \tag{53}$$

Equations Eqs. (50) and (51) are the Fokker-Planck equations corresponding, respectively, to the Langevin equations (52) and (53). Furthermore, by using the assumption (21) with Eq. (22), the above Langevin equations are written into

$$\dot{\bar{z}}(\boldsymbol{x},t)=\left(\Gamma_r\ln\frac{L}{r}\right)[-S'(\bar{z}_r)+c_r\nabla_{\boldsymbol{x}}^2\bar{z}_r]+R_r(\boldsymbol{x},t), \tag{54}$$

$$\dot{\epsilon}_r(\boldsymbol{x},t)=\left(\ln\frac{L}{r}\right)\epsilon_r\left[\left(\Gamma_r\ln\frac{L}{r}\right)\left\{-S'\left(\frac{\ln\epsilon_r/\epsilon_L}{\ln L/r}\right)+c_r\nabla_{\boldsymbol{x}}^2\frac{\ln\epsilon_r/\epsilon_L}{\ln L/r}\right\}+R_r(\boldsymbol{x},t)\right]. \tag{55}$$

These are the fundamental equations describing the spatiotemporal dynamics of exponent field and the coarse-grained energy-dissipation rate fluctuations.

IV. LANGEVIN DYNAMICS FOR EXPONENT FLUCTUATIONS AND ENERGY-DISSIPATION RATE FIELD

A. Homogeneous fluctuations

First, we consider the case when $\epsilon_r(\boldsymbol{x},t)$ does not strongly depend on the position $\boldsymbol{x}$. In this case, $\epsilon_r(\boldsymbol{x},t)$ is assumed to be spatially uniform, as the lowest approximation, and the exponent $\bar{z}_r(t)$ defined by

$$\epsilon_r(t)=\epsilon_L\left(\frac{L}{r}\right)^{\bar{z}_r(t)} \tag{56}$$

is also spatially uniform. In this case, it is sufficient to use the probability densities by

$$Q_r(z,t)=\langle\delta(\bar{z}_r(t)-z)\rangle, \tag{57}$$

$$P_r(\epsilon,t)=\langle\delta(\epsilon_r(t)-\epsilon)\rangle=\frac{\epsilon^{-1}}{\ln L/r}Q_r\left(\frac{\ln\epsilon/\epsilon_L}{\ln L/r},t\right), \tag{58}$$

instead of the forms (48) and (49).

With the above uniformity approximation on exponent field, the Langevin equations are written as

$$\dot{\bar{z}}_r(t)=-\left(\Gamma_r\ln\frac{L}{r}\right)S'(\bar{z}_r(t))+R_r(t), \tag{59}$$

$$\dot{\epsilon}_r(t)=\left(\ln\frac{L}{r}\right)\epsilon_r\left[-\left(\Gamma_r\ln\frac{L}{r}\right)S'\left(\frac{\ln\epsilon_r/\epsilon_L}{\ln L/r}\right)+R_r(t)\right], \tag{60}$$

where $S'(z)=dS(z)/dz$ and $R_r(t)$ is the Gaussian-white noise with the statistics

$$\langle R_r(t)\rangle=0,\quad\langle R_r(t)R_r(t')\rangle=2\Gamma_r\delta(t-t'). \tag{61}$$

The Fokker-Planck equation for the probability density $Q_r(z,t)$ is given by

$$\frac{\partial Q_r(z,t)}{\partial t}=\Gamma_r\frac{\partial}{\partial z}\left[Q_r^0(z)\frac{\partial}{\partial z}\left(\frac{Q_r(z,t)}{Q_r^0(z)}\right)\right]\equiv\mathcal{L}_r(z)Q_r(z,t), \tag{62}$$

where $Q_r^0(z)\propto(L/r)^{-S(z)}$ is the steady probability density for $\bar{z}_r$. The probability density $P_r(\epsilon,t)$ for $\epsilon_r(t)$ obeys

$$\frac{\partial P_r(\epsilon,t)}{\partial t}=\Gamma_r\left(\ln\frac{L}{r}\right)^2\frac{\partial}{\partial\epsilon}\left[\epsilon^2P_r^0(\epsilon)\frac{\partial}{\partial\epsilon}\left(\frac{P_r(\epsilon,t)}{P_r^0(\epsilon)}\right)\right]\equiv\mathcal{L}_r(\epsilon)P_r(\epsilon,t), \tag{63}$$

where $P_r^0(\epsilon)\propto\epsilon^{-1}(L/r)^{-S(z_r(\epsilon))}$, with $z_r(\epsilon)=\ln(\epsilon/\epsilon_L)/\ln(L/r)$ as the steady probability density.

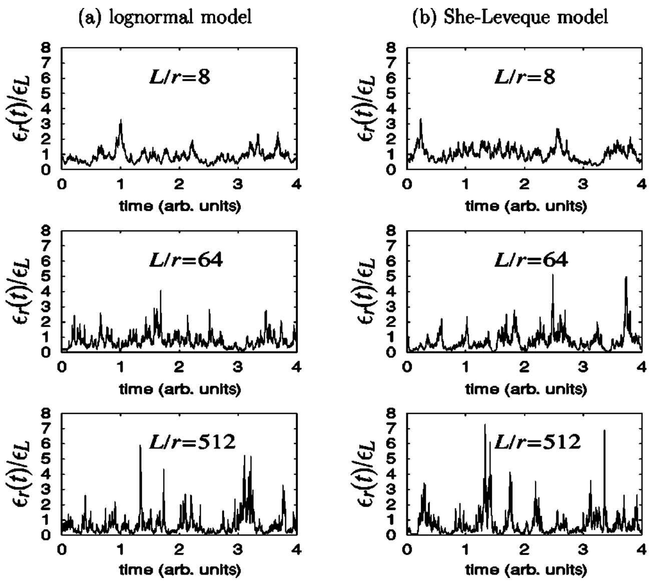

FIG. 1. Temporal evolutions of coarse-grained energy-dissipation rate $\epsilon_r(t)$ for homogeneous fluctuation for (a) the log normal model (64) with $\mu=0.22$ and (b) the She-Leveque model (65). Time evolution of $\epsilon_r(t)$ is plotted by observing that of $\overline{z_t}(t)$. For numerical integration of Eq. (59), we used the Euler method with the time increment $\Delta t=10^{-3}$, and put $\Gamma_r=1$. As the coarse-graining scale is reduced, the intermittency characteristic develops.

One specific characteristic of the equation of motion (60) for $\epsilon_r(t)$ is that the random force comes in a multiplicative way. So, for a small value of $\epsilon_r(t)$ at time t, it suddenly increases because of the concavity of $S(z)$. We carried out the time integration of Eq. (60) for the two models; (i) the log normal model [8,9],

$$S(z)=\frac{1}{2\mu}\left(z+\frac{\mu}{2}\right)^2, \tag{64}$$

with the intermittency exponent μ, and (ii) the She-Leveque model [15,17],

$$S(z)=z+\frac{\frac{2}{3}-z}{\ln\frac{3}{2}}\ln\left(\frac{\frac{2}{3}-z}{2e\,\ln\frac{3}{2}}\right). \tag{65}$$

The time integration was carried out by using the Euler method

$$\overline{z_r}(t_{j+1})=\overline{z_r}(t_j)-\left(\Gamma_r\ln\frac{L}{r}\right)S'[\overline{z_r}(t_j)]\Delta t+\sqrt{2\Gamma_r\Delta t}N(t_j), \tag{66}$$

where $t_j=j\Delta t$, $(j=1,2,3,\ldots,)$, Δt being the time increment, and $N(t_j)$ is the Gaussian noise with the zero mean $\langle N(t_j)\rangle=0$ and the correlation function $\langle N(t_j)N(t_k)\rangle=\delta_{jk}$. The time evolutions of $\epsilon_r(t)$ calculated with $\overline{z_r}(t)$ for the two models are drawn in Fig. 1. One observes that the intermittency develops as the coarse-graining scale r is reduced. As discussed above, the existence of sudden changes of $\epsilon_r(t)$ being an eminent characteristic of intermittency is the consequence of the fact that the random force comes in a multiplicative way in the time evolution of $\epsilon_r(t)$.

In order to analytically calculate the time correlation function of $\epsilon_r(t)$, we hereafter in this paper employ the Gaussian model for $Q_r^0(z)$. When $Q_r^0(t)$ is not a Gaussian, the time correlation functions can be obtained by solving the eigenvalue problem of the Fokker-Planck operator $\mathcal{L}_r(z)$ (Appendix B). Let us put

$$S(z)=\frac{1}{4D}(z-z_0)^2. \tag{67}$$

z_0 is the position of the minimum of S and D is the curvature of the function S at the minimum. This expression is valid near z_0. Since the average value of $\epsilon_r(t)=\epsilon_L(L/r)^{\overline{z_r(t)}}$ is independent of r because of the homogeneity of turbulence, we find [15]

$$z_0=-D<0. \tag{68}$$

By using the equation of motion

$$\dot{\overline{z}}_r(t)=-a_r(\overline{z_r}-z_0)+R_r(t), \tag{69}$$

$$\dot{\epsilon}_r(t)=\left(\ln\frac{L}{r}\right)\epsilon_r\left[-a_r\left(\frac{\ln\epsilon_r/\epsilon_L}{\ln L/r}-z_0\right)+R_r(t)\right], \tag{70}$$

$$a_r=\frac{\Gamma_r}{2D}\ln\frac{L}{r}, \tag{71}$$

the time correlation functions

$$C_{r,z}(t-t')=\langle\overline{z_r}(t)\overline{z_r}(t')\rangle-z_0^2, \tag{72}$$

$$C_{r,\epsilon}(t-t')=\langle\epsilon_r(t)\epsilon_r(t')\rangle-\epsilon_L^2 \tag{73}$$

are easily obtained as

$$C_{r,z}(t-t')=\frac{2D}{\ln L/r}\left(\frac{L}{r}\right)^{-(\Gamma_r/2D)|t-t'|}, \tag{74}$$

$$C_{r,\epsilon}(t-t')=\epsilon_L^2\left[\left(\frac{L}{r}\right)^{2D(L/r)^{-(\Gamma_r/2D)|t-t'|}}-1\right]. \tag{75}$$

Particularly, we get

$$C_{r,\epsilon}(t-t')=\epsilon_L^2\left[\left(\frac{L}{r}\right)^{2D(1-a_r|t-t'|)}-1\right] \tag{76}$$

for small $|t-t'|$, and the correlation function decays exponentially as

$$C_{r,\epsilon}(t-t')=\epsilon_L^2\left(2D\ln\frac{L}{r}\right)\left(\frac{L}{r}\right)^{-(\Gamma_r/2D)|t-t'|} \tag{77}$$

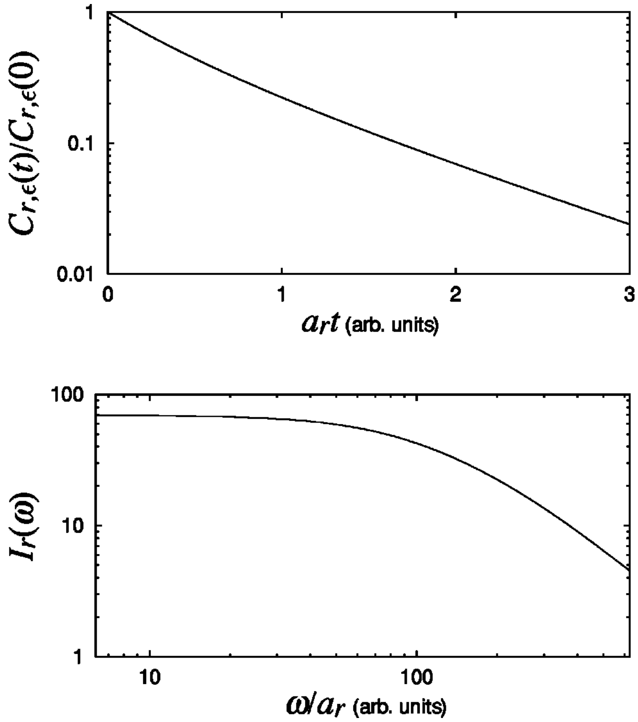

FIG. 2. (a) Time correlation function $C_{r,\epsilon}(t)$ and (b) the power spectrum of $\epsilon_r(t)$ in the homogeneous fluctuation case. Parameter values are $L/r=512$, $z_0=-D=-0.11$.

for sufficiently large $|t-t'|$. The function $C_{r,\epsilon}(t)$ and its Fourier transform $I_r(\omega)$, the power spectrum of $\epsilon_r(t)$, are shown in Fig. 2.

B. Inhomogeneous fluctuations

Next, we take into account spatially inhomogeneous fluctuations of $\overline{z(\boldsymbol{x},t)}$ and $\epsilon_r(\boldsymbol{x},t)$ with the Gaussian approximation

$$\mathcal{H}\{z\}=\int\left[\frac{1}{4D}(z(\boldsymbol{x})-z_0)^2+\frac{c_r}{2}(\nabla_{\boldsymbol{x}}z(\boldsymbol{x}))^2\right]d\boldsymbol{x}, \tag{78}$$

where $D=-z_0(>0)$. With this approximation, the Langevin equations are obtained as

$$\dot{\overline{z}}_r(\boldsymbol{x},t)=(-a_r+b_r\nabla_{\boldsymbol{x}}^2)(\overline{z}_r-z_0)+R_r(\boldsymbol{x},t), \tag{79}$$

$$\dot{\epsilon}_r(\boldsymbol{x},t)=\left(\ln\frac{L}{r}\right)\epsilon_r\left[(-a_r+b_r\nabla_{\boldsymbol{x}}^2)\left(\frac{\ln\epsilon_r/\epsilon_L}{\ln L/r}-z_0\right)+R_r(\boldsymbol{x},t)\right], \tag{80}$$

where

$$a_r=\frac{\Gamma_r}{4D}\ln\frac{L}{r},\quad b_r=c_r\Gamma_r\ln\frac{L}{r}. \tag{81}$$

With the use of the Fourier transformation

$$\psi_{\boldsymbol{k}}=\int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\psi(\boldsymbol{x})d\boldsymbol{x},\quad \psi(\boldsymbol{x})=\frac{1}{(2\pi)^3}\int e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\psi_{\boldsymbol{k}}\,d\boldsymbol{k}, \tag{82}$$

the equation of motion for exponent fluctuations is written as

$$\dot{\overline{z}}_{r,\boldsymbol{k}}(t)=-(a_r+b_rk^2)\overline{z}_{r,\boldsymbol{k}}(t)+R_{r,\boldsymbol{k}}(t). \tag{83}$$

This gives the correlation function

$$\langle[\overline{z}_r(\boldsymbol{x},t)-z_0]_{\boldsymbol{k}}[\overline{z}_r(\boldsymbol{x}',t')-z_0]_{\boldsymbol{k}'}\rangle =(2\pi)^3\frac{\Gamma_r}{a_r+b_rk^2}\delta(\boldsymbol{k}+\boldsymbol{k}')e^{-(a_r+b_r\boldsymbol{k}^2)|t-t'|}, \tag{84}$$

i.e.,

$$C_{r,z}(\boldsymbol{x}-\boldsymbol{x}',t-t') \equiv\langle\overline{z}_r(\boldsymbol{x},t)\overline{z}_r(\boldsymbol{x}',t')\rangle-z_0^2 \tag{85}$$

$$=\frac{1}{(2\pi)^3}\int\frac{\Gamma_r}{a_r+b_rk^2}e^{-(a_r+b_r\boldsymbol{k}^2)|t-t'|}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}d\boldsymbol{k}. \tag{86}$$

Furthermore, the Gaussian property of $\overline{z}_r$ enables us to find the expression of the correlation function for $\epsilon_r(\boldsymbol{x},t)$ as

$$C_{r,\epsilon}(\boldsymbol{x}-\boldsymbol{x}',t-t') \equiv\langle\epsilon_r(\boldsymbol{x},t)\epsilon_r(\boldsymbol{x}',t')\rangle-\epsilon_L^2 \tag{87}$$

$$=\epsilon_L^2\left[\exp\left\{\left(\ln\frac{L}{r}\right)^2C_{r,z}(\boldsymbol{x}-\boldsymbol{x}',t-t')\right\}-1\right]. \tag{88}$$

Particularly, one obtains

$$C_{r,z}(\boldsymbol{0},t)\propto\frac{(L/r)^{-(\Gamma_r/4D)|t|}}{|t|^{3/2}}, \tag{89}$$

$$C_{r,z}(\boldsymbol{x}-\boldsymbol{x}',0)\propto\frac{e^{-|\boldsymbol{x}-\boldsymbol{x}'|/\xi_r}}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{90}$$

where $\xi_r=\sqrt{2Dc_r}$ is the correlation length of exponent field fluctuations. We thus find that the time correlation function and the spatial correlation function decay in a power law form, respectively, for short time as $t^{-3/2}$ and for short distance as $|\boldsymbol{x}-\boldsymbol{x}'|^{-1}$, and that they decay exponentially, respectively, for long time and for long distance.

V. SUMMARY AND CONCLUDING REMARKS

In the present paper, we developed a stochastic theory of the energy-dissipation rate fluctuations in developed turbulence that are closely related to the intermittency effect on velocity structure functions and energy-dissipation rate structure functions. This was done first by proposing the static probability density for exponent field defined via coarse-grained energy-dissipation rate, constructed by phenomenologically taking into account spatial inhomogeneity of exponent field on the basis of the single-point probability density considering the asymptotic form $\propto (L/r)^{-S(z)}$. Applying the projection-operator method, we derived the Langevin dynamics for exponent field and coarse-grained energy-dissipation rate field. Furthermore, making use of the parabola approximation for the fluctuation spectrum $S(z)$, we explicitly obtained the spatiotemporal correlation functions for exponent fluctuations and coarse-grained energy-dissipation rate fluctuations for both homogeneous and inhomogeneous fluctuations. It was found that the temporal and spatial correlation functions typically decay in an exponential manner for long time and distance.

Let us add a few remarks on the present approach. As is well known in nonequilibrium statistical physics near thermal equilibrium, the projection-operator technique is quite useful to derive a closed stochastic dynamics for relevant variables under consideration. In the present paper, we tried to find a closed dynamics for the exponent field and equivalently the energy-dissipation rate fluctuations with the aid of this formalism. In order to get meaningful results, we further proposed three major approximations. The first is the use of a pure dissipative dynamics for exponent field, the second is the Fokker-Planck approximation, and the third the Markov approximation. Unfortunately, we have no solid confirmation to employ these approximations. Although these approximations make the treatment tractable, their physical fundations are not so obvious. Their validity or invalidity should be examined experimentally as well as theoretically in future. Finally, unfortunately no works on the statistical dynamics of energy-dissipation rate fluctuations are available as far as the authors know. We hope that laboratory experiments and numerical simulations provide data on them. The present results, particularly, Eqs. (89) and (90), should be compared with experimental and numerical results in future.

APPENDIX A: PROJECTION-OPERATOR DERIVATION OF EXPONENT FIELD DYNAMICS

The Langevin dynamics for $\overline{z}_r(\boldsymbol{x},t)$ can be derived in a way similar to that in deriving the macroscopic equations of motion in thermodynamic systems as in Ref. [11] developed in nonequilibrium statistical mechanics near thermal equilibrium. This is carried out as follows. By operating the identity

$$\frac{de^{tL}}{dt}=e^{tL}\mathcal{P}L+\int_0^t ds\, e^{(t-s)L}\mathcal{P}Le^{s(1-\mathcal{P})L}(1-\mathcal{P})L + e^{t(1-\mathcal{P})L}(1-\mathcal{P})L \tag{A1}$$

to $g_z(0)$, Eq. (36) is rewritten into

$$\frac{\partial g_z(t)}{\partial t}=-\int \frac{\delta}{\delta z(\boldsymbol{x})}[V_{r,\boldsymbol{x}}(z)g_z(t)]d\boldsymbol{x} + \int_0^t\int \langle LF_z(s);a\rangle g_a(t-s)da\, ds+F_z(t), \tag{A2}$$

where we defined

$$V_{r,\boldsymbol{x}}(z)=\langle L\overline{z}_r(\boldsymbol{x},0);z\rangle=\langle \dot{\overline{z}}_r(\boldsymbol{x},0);z\rangle, \tag{A3}$$

$$F_z(t)=-\int d\boldsymbol{x}\frac{\delta}{\delta z(\boldsymbol{x})}e^{t(1-\mathcal{P})L}[R_r(\boldsymbol{x},0)g_z(0)], \tag{A4}$$

$$R_r(\boldsymbol{x},0)=(1-\mathcal{P})L\overline{z}_r(\boldsymbol{x},0)=(1-\mathcal{P})\dot{\overline{z}}_r(\boldsymbol{x},0). \tag{A5}$$

Here, $\delta/\delta z(\boldsymbol{x})$ is the functional derivative, and we used the equality

$$Lg_z(0)=-\int \frac{\delta}{\delta z(\boldsymbol{x})}[\{L\overline{z}_r(\boldsymbol{x},0)\}g_z(0)]d\boldsymbol{x}. \tag{A6}$$

The quantity $V_{r,\boldsymbol{x}}(z)$ is called the streaming velocity and does not vanish provided that a collective motion in the exponent field is present [11].

The integrand of the time integration in the second term in the right hand side of Eq. (A2) is written as

$$\begin{aligned}
&\int \left[\int \{LF_z(s)\}\frac{g_a(0)}{Q_r^0\{a\}}d\mu(\boldsymbol{X})\right]g_a(t-s)da\\
&=-\int\left[\int F_z(s)Lg_a(0)d\mu(\boldsymbol{X})\right]\frac{g_a(t-s)}{Q_r^0\{a\}}da\\
&=\int\int \frac{g_a(t-s)}{Q_r^0\{a\}}\frac{\delta}{\delta a(\boldsymbol{y})}\\
&\quad\times[\langle F_z(s)\{L\overline{z}_r(\boldsymbol{y},0)\};a\rangle Q_r^0\{a\}]da\,d\boldsymbol{y}\\
&=\int\int \frac{g_a(t-s)}{Q_r^0\{a\}}\frac{\delta}{\delta a(\boldsymbol{y})}\\
&\quad\times[\langle F_z(s)R_r(\boldsymbol{y},0);a\rangle Q_r^0\{a\}]da\,d\boldsymbol{y},
\end{aligned} \tag{A7}$$

where we noticed the relation (39). Multiplying $z(\boldsymbol{x})$ to Eq. (A2) and integrating it over $z=\{z(\boldsymbol{x})\}$, one obtains

$$\dot{\overline{z}}_r(\boldsymbol{x},t)=V_{r,\boldsymbol{x}}(\overline{z}_r(t))+\int_0^t D_{r,\boldsymbol{x}}(\overline{z}_r(t-s),s)ds+R_r(\boldsymbol{x},t), \tag{A8}$$

where $\overline{z}_r(t)$ stands for the set $\{\overline{z}_r(\boldsymbol{x},t)\}$, and we have defined

$$D_{r,\boldsymbol{x}}(z,s)=\int \frac{1}{Q_r^0\{z\}}\frac{\delta}{\delta z(\boldsymbol{y})}[\langle R_r(\boldsymbol{x},s)R_r(\boldsymbol{y},0);z\rangle Q_r^0\{z\}]d\boldsymbol{y}, \tag{A9}$$

$$R_r(\boldsymbol{x},t)=e^{t(1-\mathcal{P})L}R_r(\boldsymbol{x},0). \tag{A10}$$

Equation (A8) can be regarded as the Langevin equation with the Langevin random force $R_r(\boldsymbol{x},t)$, whose temporal evolution is generated by the modified evolution operator $(1-\mathcal{P})L$ [11].

By carrying out the partial integration in Eq. (A7) with the use of the approximation

$$F_z(t) \approx -\int \frac{\delta}{\delta z(\boldsymbol{x})}[R(\boldsymbol{x},t)g_z(0)]d\boldsymbol{x}, \tag{A11}$$

Eq. (A7) is written as

$$\int \int \frac{\delta}{\delta z(\boldsymbol{x})}\left[\langle R_r(\boldsymbol{x},s)R_r(\boldsymbol{y},0);z\rangle Q_r^0\{z\}\frac{\delta}{\delta z(\boldsymbol{y})} \times\left\{\frac{g_z(t-s)}{Q_r^0\{z\}}\right\}\right]d\boldsymbol{y}\,d\boldsymbol{x}. \tag{A12}$$

The approximation (A11) makes the expansion with respect to $\delta/\delta z$ and terminates at the second order [16]. Using the approximation (A12) and assuming that no collective motion exists, which implies $V_{r,\boldsymbol{x}}(z)$ vanishes, we obtain the Langevin equation (42).

APPENDIX B: TIME CORRELATION FUNCTION FOR SPATIALLY UNIFORM FLUCTUATIONS

Let $G_1(\epsilon)$ and $G_2(\epsilon)$ are arbitrary functions of ϵ. The time correlation function $C(t)\equiv\langle G_1(\epsilon_r(t))G_2(\epsilon_r(0))\rangle$ is given by

$$C(t)=\int G_1(\epsilon)e^{t\mathcal{L}_r(\epsilon)}[P_r^0(\epsilon)G_2(\epsilon)]d\epsilon, \tag{B1}$$

where $\mathcal{L}_r(\epsilon)$ is the time evolution operator for $P_r(\epsilon,t)$. In terms of the exponent fluctuation $z=\ln(\epsilon/\epsilon_L)/\ln(L/r)$, the above is rewritten into

$$C(t)=\int G_1\left(\epsilon_L\left(\frac{L}{r}\right)^z\right)e^{t\mathcal{L}_r(z)}\left[Q_r^0(z)G_2\left(\epsilon_L\left(\frac{L}{r}\right)^z\right)\right]dz, \tag{B2}$$

where $\mathcal{L}_r(z)$ is the time evolution operator for $Q_r(z,t)$.

The eigenvalue equation for the operator $\mathcal{L}_r(z)$ is written as

$$\mathcal{L}_r(z)Q_r^\lambda(z)=-\lambda Q_r^\lambda(z), \quad (\mathrm{Re}\,\lambda\geqslant 0). \tag{B3}$$

There exists only one eigenfunction whose eigenvalue vanishes ($\lambda=0$), which is identical to the steady probability density $Q_r^0(z)$. Except this particular eigenvalue, all other eigenvalues satisfy $\mathrm{Re}\lambda>0$. By assuming the completeness of the eigenfunctions $Q_r^\lambda(z)$, and by expanding $[\cdots]$ in Eq. (B2) as

$$Q_r^0(z)G_2\left(\epsilon_L\left(\frac{L}{r}\right)^z\right)=\sum_\lambda k_r^\lambda Q_r^\lambda(z), \tag{B4}$$

with expansion coefficients k_r^λ, the correlation function is obtained as

$$C(t)=\sum_\lambda a_r^\lambda e^{-\lambda t}=\sum_{\lambda(\neq 0)} a_r^\lambda e^{-\lambda t}+\langle G_1\rangle\langle G_2\rangle, \tag{B5}$$

where

$$a_r^\lambda=k_r^\lambda\int G_1\left(\epsilon_L\left(\frac{L}{r}\right)^z\right)Q_r^\lambda(z)dz. \tag{B6}$$

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